

NuMe - SoSe23 - Homework 1

Instructions:

- Demonstration codes available in moodle can be utilized (with necessary modifications).
- The solution must be typed using LaTeX (a template `homework.tex` is available in moodle). The file can be edited online using the service <https://www.overleaf.com>.
- The submission is due May 29th 2023 at the end of the day.

Problem 1: the advection equation.

1. Using the mass flow in a control volume, derive the advection equation for a conserved quantity ρ .

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0. \quad (1)$$

2. Derive a suitable approximation of the equation, knowing that the speed, u , at which the material advects is positive.
3. For the chosen approximation, derive the order of approximation of the method.
4. For a given bounded initial value function $f(x)$, what is the solution to the advection equation? *Hint: advection is not supposed to deform the solution.*
5. Write an algorithm (step by step description of the operations) solving the advection equation up to time t_f .
6. Write the linear system solving the initial boundary value problem for Dirichlet boundary conditions.
7. Utilizing the exercise programs available (python notebooks), implement your algorithm for an arbitrary bounded initial value function $f(x)$ with Dirichlet boundary conditions

$$\rho_0(0) = f(0). \quad (2)$$

Demonstrate the results of your algorithm with figures.

8. From the numerical solution of your algorithm, compute the error for increasing resolution $\Delta x \rightarrow 0$, and measure the order of approximation. *See [?] for how to normalize the error in grid functions.*

Problem 2: the heat & Poisson equations.

1. Using the mass flow in a control volume, derive the heat equation for a the temperature T .

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}. \quad (3)$$

2. For fixed Dirichlet (and/or wall) boundary conditions, what happens to the change in time of the temperature when $t \rightarrow \infty$? *Hint: elliptic equation.*
3. Utilizing the result to the previous question, how can we indirectly solve the Poisson equation?
4. Derive a suitable approximation of the equation, utilizing the idea that diffusion is not directional.
5. For the chosen approximation, derive the order of approximation of the method.
6. Write an algorithm (step by step description of the operations) solving the Poisson equation up to tolerance $1e - 7$.
7. Write the linear system solving the initial boundary value problem for Dirichlet boundary conditions.
8. Utilizing the exercise programs available (python notebooks), implement your algorithm for an arbitrary bounded initial value function $f(x)$ with Neumann boundary conditions

$$\frac{\partial T_{\text{wall}}}{\partial x} = 0. \quad (4)$$

Demonstrate the results of your algorithm with figures.

9. Can the algorithm be used to solve the heat equation instead? If so, what component of your algorithm can be changed to get a more efficient algorithm?

References

- [1] Randall J LeVeque. *Finite difference methods for differential equations*, volume 585. 1998.