

NuMe - SoSe23 - Homework 2

Instructions:

- Demonstration codes available in moodle can be utilized (with necessary modifications).
- The solution must be typed using LaTeX (a template `homework.tex` is available in moodle). The file can be edited online using the service <https://www.overleaf.com>.
- The submission is due June 30th 2023 at the end of the day.

Problem 1: the finite volumes method.

1. For a system of conservation law

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0, \quad (1)$$

derive its finite volumes conservative form in a generic cell of width Δx .

2. Apply the above derivation to the advection equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0, \quad (2)$$

what is the flux $f(\rho)$ in this system?

3. Use the upwind flux

$$\hat{f}_{i+\frac{1}{2}} = u \frac{\rho_{i+1} + \rho_{i-1}}{2} - |u| \frac{\rho_{i+1} - \rho_i}{2}, \quad (3)$$

in the finite volumes conservative form to derive the upwind scheme. Which part of the scheme is diffusive? Was there diffusion in the original advection equation? What effect does this have on the solution?

4. For the Lax-Fredrichs scheme

$$q_i^{n+1} = \frac{q_{i-1}^n + q_{i+1}^n}{2} - \frac{\Delta t}{\Delta x} \frac{f_{i+1} - f_{i-1}}{2}, \quad (4)$$

rewrite the system in the finite volumes conservative form. What is the form of the flux function $\hat{f}_{i+\frac{1}{2}}$ for this scheme? *The time derivative was discretized using the forward Euler explicit step*

Problem 2: the momentum equation.

1. Using the momentum in a control volume, derive the momentum equation for the conserved quantity ρu_i .

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right). \quad (5)$$

Deriving the stress terms on the control volume boundary is not required, only the derivation of their contribution to the momentum equation is.

2. From the equation above, and the knowledge of the terms on the boundary of the control volume, what boundary conditions are needed to close this system?
3. Derive a suitable approximation of the equation using the finite volumes method. *The choice of the numerical flux is not important here and can be written implicitly.*
4. From your previous answer, which term cannot be computed (or updated) by approximating the derivatives directly? Why? *Density ρ can be ignored here.*
5. For the system with the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \quad (6)$$

and the momentum equation, simplify the system when ρ is constant. *incompressible Navier–Stokes.*

6. From the simplified system above, and knowing that the second-derivatives of p , and the third derivatives of u exist, derive an equation that enables us to solve for the term we were unable to compute before?
7. From the equation derived in the previous question, which boundary condition can we choose for the pressure other than what you answered before?
8. Using the forward Euler explicit time step, write an algorithm (step by step description of the operations) solving the incompressible Navier–Stokes. *Pressure solver idea.*