

Finite Volumes Method: concepts of momentum

Discretizations of incompressible flow

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Conservation laws

the Cauchy problem

For a quantity (or set of quantities) q ,

$$\frac{\partial q}{\partial t} + \frac{\partial f_j(q)}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\epsilon_{ij} \frac{\partial q}{\partial x_i} \right), \quad (1)$$

with appropriate boundary conditions (Dirichlet/Neumann or mixed, Robin)

$$q(x_\Gamma, t) = g(t) \qquad \frac{\partial q(x_\Gamma, t)}{\partial \hat{n}} = h(t), \quad (2)$$

and initial condition $q(x, 0) = q_0(x)$. Equations 1-3 define a parabolic system for positive ϵ_{ij} .

Conservation laws

the Cauchy problem for vanishing viscosity

When the viscosity coefficient is taken with a limit to zero. The system reduces to a first-order hyperbolic conservation law

$$\frac{\partial q}{\partial t} + \frac{\partial f_j(q)}{\partial x_j} = 0, \quad (3)$$

with appropriate Dirichlet boundary conditions

$$q(x_\Gamma, t) = g(t) \quad (4)$$

and initial condition $q(x, 0) = q_0(x)$.

Conservation laws: the Cauchy problem for vanishing viscosity

The advection equation

For a quantity ρ . The advection equation is a case when $f_j(\rho) = u_j \rho$, for constant u_j , giving the system

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = 0, \quad (5)$$

with appropriate Dirichlet boundary conditions

$$\rho(x_\Gamma, t) = g(t) \quad (6)$$

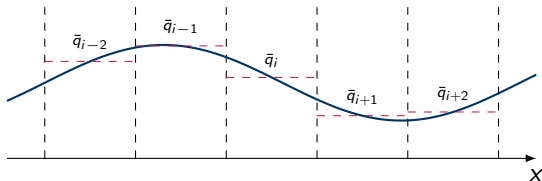
and initial condition $\rho(x, 0) = \rho_0(x)$.

Finite Volumes Method

the Cell average

For a discretized domain into sets of volumes, areas, or lengths, V_i , depending on the dimension of the problem. A cell average is chosen to represent the continuous values of the underlying solution in the cell

$$\bar{q}_i = \frac{1}{V_i} \int_{V_i} q \, dV_i. \quad (7)$$



Finite Volumes Method: the cell average

Applying the cell average to the Cauchy problem,

$$\frac{\partial \bar{q}}{\partial t} + \frac{1}{V_i} \int_{V_i} \frac{\partial f_j^{\text{inviscid}}(q)}{\partial x_j} dV_i = \frac{1}{V_i} \int_{V_i} \frac{\partial f_j^{\text{viscous}}(q)}{\partial x_j} dV_i. \quad (8)$$

We keep the terms in the integral form for further derivation.

Simplification

To simplify the derivation, we focus only on the convective part, without loss of generality,

$$\frac{\partial \bar{q}}{\partial t} + \frac{1}{V_i} \int_{V_i} \frac{\partial f_j(q)}{\partial x_j} dV_i = 0. \quad (9)$$

Finite Volumes Method: face values

We use an old trick (theorem) relating derivatives of known functions to their value at the boundary. What could it be?

Finite Volumes Method: face values

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The fundamental theorem of Calculus

For a continuous function $f(x)$ defined in a closed interval, $[x_a, x_b]$,

$$\int_{x_a}^{x_b} \frac{\partial f(x)}{\partial x} dx = f(x_b) - f(x_a). \quad (10)$$

Finite Volumes Method: face values

Practical use

In a structured grid, $V_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, centered at $x_i = x_0 + i\Delta x$

$$\frac{1}{V_i} \int_{V_i} \frac{\partial f(x)}{\partial x} dx = \frac{1}{V_i} \left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right). \quad (11)$$

Notation

The notation $\cdot_{i-\frac{1}{2}}$ and $\cdot_{i+\frac{1}{2}}$ represent the left and right faces, respectively.

$$x_{i\pm\frac{1}{2}} = x_0 + \left(i \pm \frac{1}{2}\right)\Delta x = x_i \pm \frac{1}{2}\Delta x \quad (12)$$

Finite Volumes Method

Definition: the finite volumes method

Plugging (12) back into (10) yields the finite volume method (in conservative form) for a uniform grid in one dimension

$$\frac{\partial \bar{q}}{\partial t} = -\frac{1}{V_i} \left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right). \quad (13)$$

Finite Volumes Method: approaches

The method relies on values that lie on cell centers, $\cdot_i$, and vertex values $\cdot_{i\pm\frac{1}{2}}$. Defining two approaches:

Cell centered approach:

- 1 given $\cdot_i$
- 2 interpolate to get $\cdot_{i\pm\frac{1}{2}}$
- 3 update $\cdot_i$

Vertex valued approach:

- 1 given $\cdot_{i\pm\frac{1}{2}}$
- 2 interpolate to get $\cdot_i$
- 3 update $\cdot_{i\pm\frac{1}{2}}$

Finite Volumes Method

interpolation schemes (information flux)

Focusing on the cell centered approach, we need to compute the face values via interpolation, defining interpolation schemes. These schemes should be

- 1 consistent (represent the original flux when q is constant)

$$f_{i\pm\frac{1}{2}}(q_{i-p/2}, \dots, q_i, \dots, q_{i+p/2}) = f_{i\pm\frac{1}{2}}(\bar{q}, \dots, \bar{q}) = f(\bar{q}) \quad (14)$$

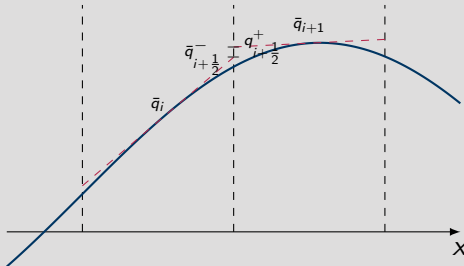
- 2 accurate (represents the correct derivative when $V_i \rightarrow 0$)

$$\frac{1}{V_i} \left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) = \frac{\partial f(q)}{\partial x} + \mathcal{O}(\Delta x^p) \quad (15)$$

Finite Volumes Method

averaging (numerical dissipation)

Because the interpolation scheme may not be exact, i.e., the interpolated value, q^- , from the current (left) cell may not match the interpolated value, q^+ , from the next (right) cell.



Finite Volumes Method

averaging (numerical dissipation)

Thus, a controller of the jump (difference) between the interpolated values is added to smoothen (dissipate) the jump

$$f_{i \pm \frac{1}{2}}^{\text{dissipation}} = C (q^+ - q^-). \quad (16)$$

This is not physical! But, it is necessary to ensure stability. This idea is equivalent to averaging the cell center over additional neighboring cells.

Finite Volumes Method

a Finite Volumes scheme

Combining the interpolation and dissipation gives a finite volume scheme

$$\frac{\partial \bar{q}}{\partial t} = -\frac{1}{V_i} \left(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}} \right), \quad (17)$$

with the combined flux function

$$\hat{f} = f^{\text{interpolation}} - f^{\text{dissipation}} \quad (18)$$

Finite Volumes Method

Interpolation:

- flow of information
- shapens the solution
- produces instabilities

Dissipation:

- controls the jump
- smoothes the solution
- reduces instabilities

Note

Smoothing the solution here is a non-physical behaviour and leads to loss of information. Schemes are designed to optimize for maximum stability and minimum loss of information. At the limit of $V_i \rightarrow 0$, the numerical dissipation vanishes, and are less impactful for finer grids.

Finite Volumes Method: example schemes

Upwind

$$\hat{f}_{i+\frac{1}{2}}(q_i, q_{i+1}) = \frac{f_{i+1} + f_i}{2} - |\lambda| \frac{q_{i+1} - q_i}{2}. \quad (19)$$

In general, λ represents the speed at which information flows, we can generalize it to $\lambda = \frac{df}{dq}$. In the case of the advection equation, it is simply u .

Finite Volumes Method: example schemes

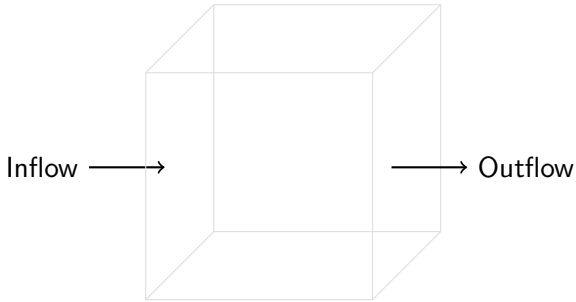
Upwind dissipation

By accounting only for the dissipation part in equation (14), we have a dissipation term

$$\left(\frac{|\lambda| \Delta x}{2} \right) \frac{q_{i+1} - 2q_i + q_{i-1}}{\Delta x^2} = \left(\frac{|\lambda| \Delta x}{2} \right) \frac{\partial^2 q}{\partial x^2} + \mathcal{O}(\Delta x^3), \quad (20)$$

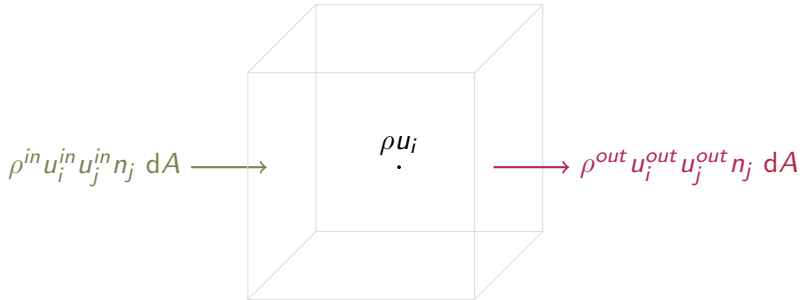
a diffusion term with a coefficient that depends on the grid.

Momentum in a fixed control volume: convective term



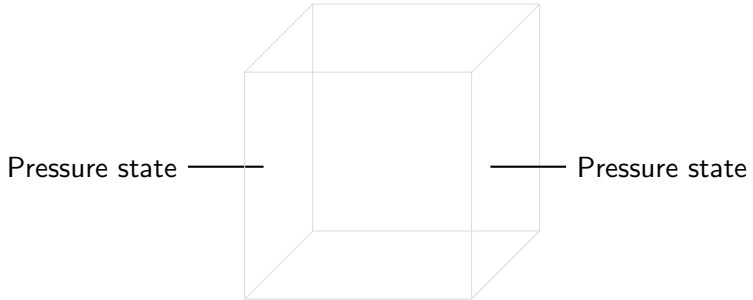
Change over time = Inflow – Outflow + Internal forces + Shear forces + External forces.

Momentum in a fixed control volume: convective term



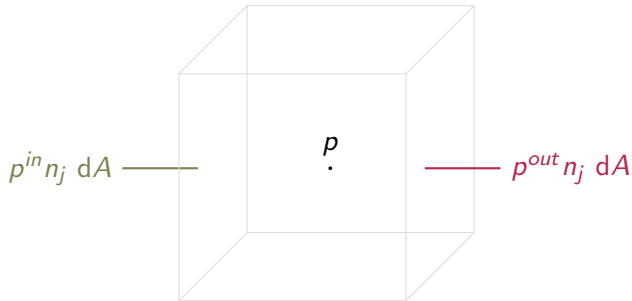
$$\int_V \frac{\partial \rho u_i}{\partial t} dV = \int_A \rho^{in} u_i^{in} u_j^{in} n_j dA - \int_A \rho^{out} u_i^{out} u_j^{out} n_j dA + \text{Internal forces} + \text{Shear forces} + \text{External forces.} \quad (21)$$

Momentum in a fixed control volume: pressure difference



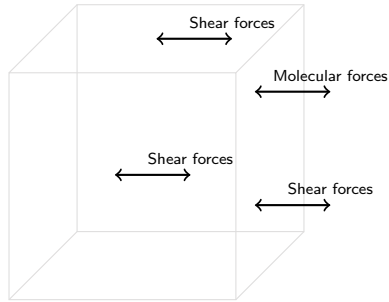
Change over time = Inflow – Outflow + Molecular forces + Shear forces + External forces.

Momentum in a fixed control volume: pressure difference



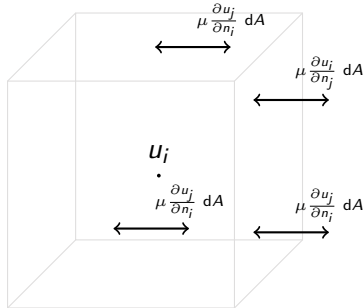
$$\int_V \frac{\partial \rho u_i}{\partial t} dV = \text{Convection} - \int_A p^{in} n_j dA + \int_A p^{out} n_j dA + \text{Shear forces} + \text{External forces.} \quad (22)$$

Momentum in a fixed control volume: stress terms



Change over time = Inflow – Outflow + Molecular forces + Shear forces + External forces.

Momentum in a fixed control volume: stress terms



$$\int_V \frac{\partial \rho u_i}{\partial t} dV = \text{Convection} - \text{Pressure} + \int_A \mu \left(\frac{\partial u_i^f}{\partial n_j} + \frac{\partial u_j^f}{\partial n_i} \right) dA + \text{External forces.} \quad (23)$$

Momentum in a fixed control volume

- We know ρ and u at the center, how do we get the values at the faces?

Momentum in a fixed control volume

- We know ρ and u at the center, how do we get the values at the faces?
 - Again... using a Taylor expansion around the cell center

The Taylor approximation around the point at the face

Let the control volume be defined with side length dx , and a point x as its center. The face where the inflow is coming from, for example, is at $x - \frac{1}{2}dx$. We can write the values at the faces as

$$\rho^{in} u_i^{in} u_j^{in} = \rho u_i u_j - \frac{1}{2} dx \frac{\partial \rho u_i u_j}{\partial x} + \frac{1}{4} dx^2 \frac{\partial^2 \rho u_i u_j}{\partial x^2} + \dots \quad (24)$$

$$\rho^{out} u_i^{out} u_j^{out} = \rho u_i u_j + \frac{1}{2} dx \frac{\partial \rho u_i u_j}{\partial x} + \frac{1}{4} dx^2 \frac{\partial^2 \rho u_i u_j}{\partial x^2} + \dots \quad (25)$$

Momentum in a fixed control volume

Inserting the computed values into the conservation equation

$$\begin{aligned}
 \int_V \frac{\partial \rho u_i}{\partial t} dV &= \int_A \rho u_i u_j n_j - \frac{1}{2} dx \frac{\partial \rho u_i u_j}{\partial x_j} n_j + \frac{1}{4} dx^2 \frac{\partial^2 \rho u_i u_j}{\partial x_j^2} n_j + \dots dA \\
 &\quad - \int_A \rho u_i u_j n_j + \frac{1}{2} dx \frac{\partial \rho u_i u_j}{\partial x_j} n_j + \frac{1}{4} dx^2 \frac{\partial^2 \rho u_i u_j}{\partial x_j^2} n_j + \dots dA \\
 &\quad - \int_A p^{f_i} n_i dA + \int_A \mu \left(\frac{\partial u_i^f}{\partial n_j} + \frac{\partial u_j^f}{\partial n_i} \right) dA + \int_V F dV. \\
 &= - \int_A \frac{\partial \rho u_i u_j}{\partial x_j} n_j dx_j dA - \int_A \frac{\partial p}{\partial x_i} n_i dx_j dA + \int_A \frac{\partial}{\partial x_j} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) dx_j dA + \int_V F dV. \\
 &= - \int_V \frac{\partial \rho u_i}{\partial x_j} dV - \int_V \frac{\partial p}{\partial x_i} dV + \int_A \frac{\partial}{\partial x_j} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) dV + \int_V F dV.
 \end{aligned}
 \tag{26}$$

Note

We applied the same technique to obtain the pressure gradient and stress terms.

Momentum in a fixed control volume

- We know ρ and u at the center, how do we get the values at the faces?
 - Using a Taylor expansion around the cell center
- The same derivation is valid for pressure, viscous, and shear forces on opposite faces.

The momentum equation

Taking the limit $dV \rightarrow 0$ leads to the following momentum equation

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) + F. \quad (27)$$

Incompressible flows

the Navier–Stokes equation

Combining the continuity equation and the momentum equation gives the Navier–Stokes system

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0 \quad (28)$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) + F \quad (29)$$

Incompressible flows: divergence free condition

Divergence free condition

Incompressible flows are described with a constant density. Assuming constant density leads equation (28) to become

$$\frac{\partial u_j}{\partial x_j} = 0. \quad (30)$$

Incompressible flows: simplified momentum

Momentum equation for incompressible flows

Constant density and the divergence free condition simplifies the momentum equation (29) to

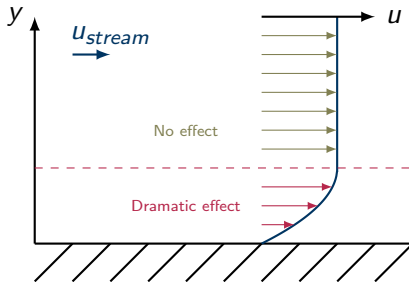
$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F. \quad (31)$$

Laminar boundary layers

Solid wall boundary conditions

Relevant to viscous flows, we study of the effect of a solid wall on passing flow. At the solid wall, the velocity is zero, and far away from it, the velocity is constant (assuming the forces are balanced there and the flow is laminar).

Laminar boundary layers



Boundary layer

The boundary layer is defined as the area where the effect of the wall boundary condition causes a dramatic change in the velocity of the fluid as we approach the wall. Its height is dictated by the viscosity of the fluid.

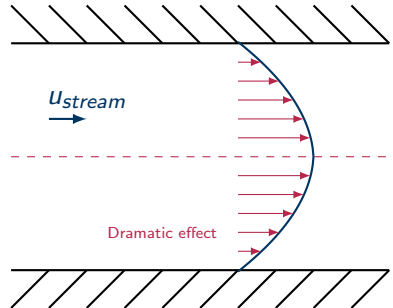
Laminar boundary layers: examples of internal flows

Flow in a pipe

Assuming the pipe is infinitely long, the system (30-31) reduces to an ODE for the change in the stream-wise velocity along the radius of the channel. The steady state solution is given by

$$u(y) = \frac{1}{\mu} \frac{\partial p}{\partial x} f(y, z), \quad (32)$$

where $f(y, z)$ is a function that depends on the geometry.



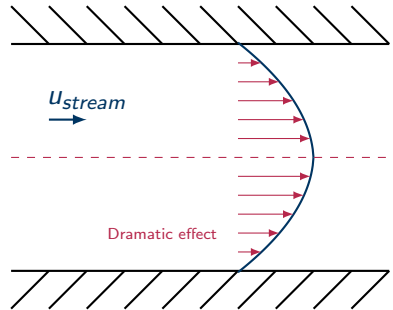
Laminar boundary layers: examples of internal flows

Flow between two flat sheets

The flow between two flat sheets is one of the classic examples of laminar boundary layers. In this example, because the channel is infinitely wide, the solution is

$$f(y) = \frac{1}{2}(R - r(y)^2), \quad (33)$$

where R is the radius (or half height) of the channel, and $r(y)$ is radial position.

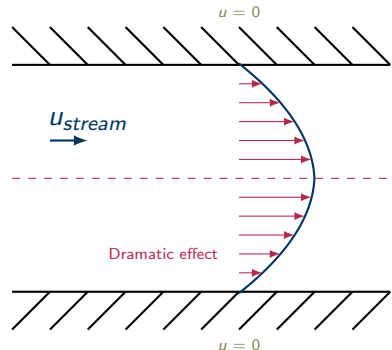


Laminar boundary layers: examples of internal flows

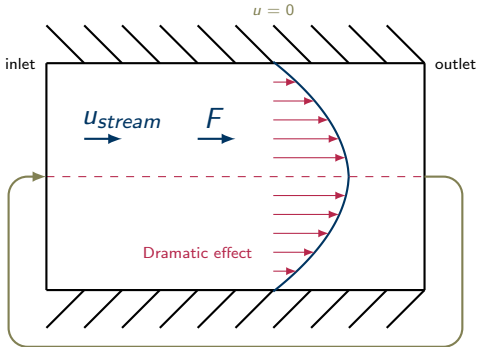
Flow between two flat sheets

To simulate this numerically, we start from the system (30-31). By controlling the boundary conditions, we impose the assumptions numerically, and let the flow converge to the steady state solution.

The most important boundary condition in this case is the no-slip wall boundary condition, where $u = 0$ at the wall.



Laminar boundary layers: examples of internal flows



Flow between two flat sheets

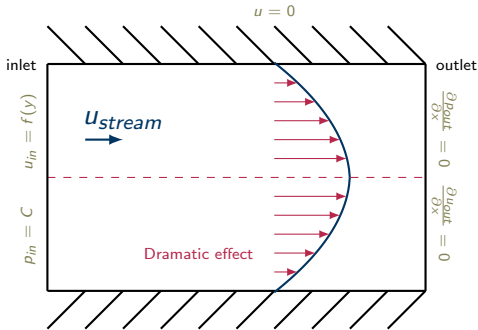
The simplest inlet/outlet setup is periodic boundary conditions

$$u_{out} = u_{in}, \quad (34)$$

$$p_{out} = p_{in}. \quad (35)$$

This assumes that the pipe is infinitely long. However, that means we have to force the flow via a source term, F .

Laminar boundary layers: examples of internal flows



Flow between two flat sheets

Another option is to impose a state at the inlet

$$u_{in} = f(y) \quad (36)$$

$$p_{in} = C, \quad (37)$$

and either a state, or a condition that ensures no inflow comes from the outlet. Typically, gradient-zero

$$\frac{\partial u_{out}}{\partial x} = 0, \quad \frac{\partial p_{out}}{\partial x} = 0. \quad (38)$$

Solution strategies

Underdetermined system

The system (28-29) is underdetermined, containing 3 unknowns, ρ , u , p , and only 2 equations. To alleviate this, we do one of the following:

- introduce another equation to complete the system
- utilize assumptions to reduce the number of variables

Solution strategies: compressible flow

Density based solution

For compressible flows (non-constant density), the temperature is coupled with the system, introducing another variable, T , which is solved for, typically, by the energy equation (not discussed further),

$$\frac{\partial \rho \mathcal{E}}{\partial t} + \frac{\partial}{\partial x_j} (u_i (\rho \mathcal{E} + p)) = \frac{\partial}{\partial x_j} (\mu u_i \tau_{ij}) + \frac{\partial}{\partial x_j} \left(\kappa \frac{\partial T}{\partial x_j} \right), \quad (39)$$

where κ is the heat conductivity. Then, the variables are coupled via an equation of state. For example, for ideal gas

$$p = \rho R T, \quad (40)$$

where R is the gas constant.

Solution strategies: compressible flow

Compressible flow

The system (28-29-39-40) is fully determined. For high-speed flows (typically, with Mach number $Ma = u_{\text{stream}}/c > 0.3$, where c is the speed of sound), the system is “more hyperbolic,” and thus is treated as a hyperbolic conservation law.

Density solver

An algorithm solving such a system by discretizing the spatial part directly is called a density solver.

Solution strategies: compressible flow

Density solver use cases

A density solver is used to solve for compressible flows. However, it can be used to solve for incompressible flows (low Mach) if conditioned correctly (not discussed here).

Solution strategies: incompressible flow

How to update the pressure?

For incompressible flows, the system (30-31) is determined for the variables u and p . However, a direct discretization of the spatial part is not possible, because only the velocity is updated in the time step. Essentially, we cannot update the pressure this way. So, what do we do?

Solution strategies: incompressible flow

What do we do? Pressure Solver idea

Taking the divergence of the momentum equation gives

$$\frac{\partial}{\partial t} \frac{\partial u_i}{\partial x_i} + \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\nu \frac{\partial^2 u_i}{\partial x_j^2} \right) + \frac{\partial F}{\partial x_i}. \quad (41)$$

Assuming the divergence free condition gives an elliptic equation for the pressure

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left(-u_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F \right) = \frac{\partial H_i}{\partial x_i}. \quad (42)$$

Solution strategies

What do we do? Pressure Solver idea

If at a given time, we know H_i , but $\frac{\partial u_i}{\partial x_i} \neq 0$, solving for p in

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i^2} = \frac{\partial H_i}{\partial x_i}, \quad (43)$$

implies

$$\frac{\partial}{\partial t} \frac{\partial u_i}{\partial x_i} = -\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i^2} + \frac{\partial H_i}{\partial x_i} = 0. \quad (44)$$

Thus, by solving the elliptic equation for the pressure, we enforce the divergence free condition!

Solution strategies

What do we do? Pressure Solver base algorithm

1 given H_i , but $\frac{\partial u_i}{\partial x_i} \neq 0$

2 solve for p in

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i^2} = \frac{\partial H_i}{\partial x_i}, \quad (45)$$

such that $\frac{\partial u_i}{\partial x_i} = 0$

3 solve for u_i in

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F. \quad (46)$$

Literature

- Masatsuka, Katate. "I do like CFD, VOL. 1, version 2.6." (2018). pp. 45-46
- LeVeque, R. hyper. cons.

Summary of Concepts

Equations of motion

Equation	Form	Examples
Mass flow	$\frac{\partial \psi}{\partial t} + \frac{\partial f(\psi)}{\partial x} = 0$	Advection, Burgers, Continuity
Momentum	$\frac{\partial \rho \psi}{\partial t} + \frac{\partial \rho \psi u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} (\mu \tau_{ij}) + F$	Navier–Stokes
Energy		

Summary of concepts

Types of PDEs

Type	Form	Examples
Hyperbolic	$\frac{\partial^2 \psi}{\partial t^2} - w^2 \frac{\partial^2 \psi}{\partial x^2} = 0$	Advection, Burgers
Parabolic	$\frac{\partial \psi}{\partial t} + \frac{\partial f(\psi)}{\partial x} = \epsilon \frac{\partial^2 \psi}{\partial x^2}$	Heat
Elliptic	$\epsilon \frac{\partial^2 \psi}{\partial x^2} = 0$	Heat (at $t \rightarrow \infty$), Pressure term (in NS)

Thanks for your attention!

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